

## IV<sup>th</sup> - Laplace & z-transforms.

①

- ⇒ Laplace transform represents continuous time signal in terms of complex exponential i.e. " $e^{-st}$ ".
- ⇒ It is used to analyze the signals or function which are not absolutely integrable.
- ⇒ More effectively, continuous time signals can be analyzed using Laplace transform.
- ⇒ Laplace transform provides broader characterization. Compare to Fourier transform.

### Definition :

To transform a time domain signal  $x(t)$  to s-domain multiply the signal by " $e^{-st}$ " and then integrate " $-\infty$ " to " $\infty$ ".

⇒ The transformed represented as  $X(s)$  and transformation is denoted by " $L$ ".

⇒ Laplace transform is given for continuous time signal  $x(t)$

$$\text{i.e. } X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt \rightarrow \textcircled{1}$$

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⇒ Where "s" is complex in nature and given as  $s = \sigma + j\omega$

Where " $\sigma$ " real part (or) attenuation constant

Where " $j\omega$ " is imaginary (or) complex frequency

⇒ If  $x(t)$  is defined for  $t \geq 0$  (i.e. casual system (or)  $x(t)$  casual) then Laplace transform of  $x(t)$

$$L\{x(t)\} = X(s) = \int_0^{\infty} x(t) \cdot e^{-st} \cdot dt \rightarrow \textcircled{2}$$

Types of Laplace transform:

⇒ It broadly classified into two categories

(i) Bi-lateral- (or) two sided transform

(ii) Uni-lateral (or) one sided transform.

(i) Bi-lateral (or) two sided:

If integration taken from  $-\infty$  to  $+\infty$ , as shown in eq'n ①. then it is called bi-lateral Laplace transform.

Uni-lateral (or) one sided :

If integration taken from "0" to " $\infty$ ", as show in eq'n ②. Then it is called Uni-lateral Laplace transform

Inverse Laplace transform:

$\Rightarrow$  The s-domain signal  $x(s)$  can be transformed to time domain signal  $x(t)$ . By using inverse Laplace transform.

$\Rightarrow$  And is defined as

$$\mathcal{L}^{-1} \{ x(s) \} = x(t) = \frac{1}{2\pi j} \int_{-j\omega}^{+j\omega} x(s) e^{-st} \cdot ds$$

$\Rightarrow$  The signal  $x(s)$  &  $x(t)$  are called Laplace transform pair.

i.e

$$x(t) \xleftrightarrow{\text{LT}} x(s)$$

$$x(s) \xleftrightarrow{\text{ILT}} x(t)$$

Relation between Fourier transform & Laplace transform

$\Rightarrow$  Fourier transform is given as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \cdot dt \longrightarrow \text{①}$$

Fourier transform can be calculated only if  $x(t)$  is absolutely integrable

$$\text{i.e. } \int_{-\infty}^{\infty} |x(t)| \cdot dt < \infty \rightarrow \textcircled{2}$$

Laplace transform is written as

$$X(s) = \int_{-\infty}^{\infty} e^{-st} \cdot x(t) \cdot dt$$

putting  $s = \sigma + j\omega$

The above eq'n

$$X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-(\sigma + j\omega)t} \cdot dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot e^{-\sigma t} \cdot e^{-j\omega t} \cdot dt$$

$$= \left\{ \int_{-\infty}^{\infty} x(t) \cdot e^{-t} \right\} e^{-j\omega t} \cdot dt \rightarrow \textcircled{3}$$

compare eq'n  $\textcircled{3}$  with eq'n  $\textcircled{2}$  Laplace transform of  $x(t)$  is basically a Fourier transform of

$$x(t) \cdot e^{-\sigma t}$$

If  $\sigma = 0$ , then eq'n  $s = j\omega$

$$\therefore X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt = X(j\omega) / \text{when } \sigma = 0 \text{ (or, } s = j\omega)$$

It basically a fourier transform on imaginary  $j\omega$  axis in  $s$ -plane.

Convergence/Region of convergence (Roc):

$$\text{From equation } \left\{ \int_{-\infty}^{\infty} [x(t) \cdot e^{-\sigma t}] e^{-j\omega t} \cdot dt \right\}$$

Wki, Laplace transform is a basically the fourier transform  $x(t) \cdot e^{-\sigma t}$ . If fourier transform  $x(t) \cdot e^{-\sigma t}$  exists then Laplace transform  $x(t)$  is also exists.

$$\Rightarrow \int_{-\infty}^{\infty} |x(t) \cdot e^{-\sigma t}| \cdot dt < \infty,$$

must be absolutely integrable for fourier transform to exists

$\Rightarrow$  Laplace transform of  $x(t)$  is exists if above condition satisfy.

$\Rightarrow$  The range of values ' $\sigma$ ' for which Laplace transform converges is called Roc.

(or),

⇒ The Laplace transform of signals is given by

$$\int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

The above values of "s" for which the integral

-L

" $\int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$ " converges is called Region of

Convergence

properties of Laplace transform:

(1) Amplitude scaling:

If Laplace transform of

$$\text{if } L\{x(t)\} = X(s)$$

then

$$L[A \cdot x(t)] = A X(s)$$

proof:

W.K.T,

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

$$L[A x(t)] = \int_{-\infty}^{\infty} A \cdot x(t) \cdot e^{-st} \cdot dt$$

$$= A \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

$$L [x(t)] = A x(s)$$

(2) Linearity:

Let,

$$x(t) = ax_1(t) + bx_2(t)$$

$$L [x(t)] = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

$$= L [ax_1(t)] = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

$$= L [ax_1(t)] = ax_1(s) \quad \& \quad L [bx_2(t)] = bx_2(s)$$

$$= L [ax_1(t) + bx_2(t)] = ax_1(s) + bx_2(s)$$

proof:

$$L [x(t)] = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

$$= \int_{-\infty}^{\infty} (ax_1(t) + bx_2(t)) \cdot e^{-st} \cdot dt$$

$$= \int_{-\infty}^{\infty} ax_1(t) \cdot e^{-st} + \int_{-\infty}^{\infty} bx_2(t) \cdot e^{-st} \cdot dt$$

$$= ax_1(s) + bx_2(s)$$

$$\therefore L [ax_1(t) + bx_2(t)] = ax_1(s) + bx_2(s)$$

(3) time differentiation: {Consider the casual condition}

$$\text{if } L[x(t)] = X(s)$$

then,

$$L\left[\frac{d}{dt} x(t)\right] = sX(s) - x(0)$$

Where  $x(0)$  is the value of  $x(t)$  at  $t=0$ .

Proof:

$$X(s) = L[x(t)] = \int_0^{\infty} x(t) \cdot e^{-st} \cdot dt$$

$$L\left[\frac{d}{dt} x(t)\right] = \int_0^{\infty} \frac{d}{dt} x(t) \cdot e^{-st} \cdot dt$$

$$= \int_0^{\infty} e^{-st} \cdot dt$$

$$= e^{-st} \int_0^{\infty} \frac{d}{dt} x(t) - \int_0^{\infty} \left[ -s e^{-st} \int_0^{\infty} \frac{d}{dt} x(t) \cdot dt \right]$$

$$= \left[ e^{-st} x(t) \right]_0^{\infty} - \int_0^{\infty} -s e^{-st} \cdot x(t) \cdot dt$$

$$= sX(s) - x(0).$$

(4) Time integration: [Consider Uni-lateral]

if  $L[x(t)] = X(s)$

then,

$$L\left\{\int x(t) \cdot dt\right\} = \frac{X(s)}{s} + \frac{\left[\int x(t) \cdot dt\right]}{s} \Big|_{t=0}$$

Proof:

$$X(s) = L\left\{x(t)\right\} = \int_0^{\infty} x(t) \cdot e^{-st} \cdot dt$$

$$L\left\{\int x(t) \cdot dt\right\} = \int_0^{\infty} \left[\int x(t) \cdot dt\right] e^{-st} \cdot dt$$

$$= \left[ \left[\int x(t) \cdot dt\right] \cdot \frac{e^{-st}}{-s} \right]_0^{\infty} - \int_0^{\infty} x(t) \frac{e^{-st}}{-s} \cdot dt$$

$$= \left[ \int x(t) \cdot dt \right]_{t=\infty} \frac{e^{-\infty}}{-s} - \left[ \int x(t) \cdot dt \right] \Big|_{t=0} \frac{e^0}{-s} + \frac{1}{s} \int_0^{\infty} x(t) \cdot e^{-st} \cdot dt$$

$$= \frac{1}{s} \left[ \int x(t) \cdot dt \right] \Big|_{t=0} + \frac{1}{s} \int_0^{\infty} x(t) \cdot e^{-st} \cdot dt$$

$$L\left\{\int x(t) \cdot dt\right\} = \frac{X(s)}{s} + \frac{\left[\int x(t) \cdot dt\right]}{s} \Big|_{t=0}$$

### ⑤ Frequency shifting:

If  $L\{x(t)\} = X(s)$ , then

$$L\{e^{\pm at} x(t)\} = X(s \pm a)$$

Proof:

$$\begin{aligned} X(s) &= L\{x(t)\} = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt \\ &= \int_{-\infty}^{\infty} (e^{\pm at} \cdot x(t)) \cdot e^{-st} \cdot dt \\ &= \int_{-\infty}^{\infty} x(t) \cdot e^{-(s \pm a)t} \cdot dt \end{aligned}$$

$$\therefore L\{e^{\pm at} \cdot x(t)\} = X(s \pm a)$$

### ⑥ Time shifting:

If  $L[x(t)] = X(s)$  then,

$$L\{x(t \pm a)\} = X(s) \cdot e^{\pm as}$$

Proof:

$$\begin{aligned} X(s) &= L\{x(t)\} = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt \\ &= \int_{-\infty}^{\infty} x(t \pm a) \cdot e^{-st} \cdot dt \end{aligned}$$

put

$$t \pm a = z$$

$$t = z \mp a$$

$$dt = dz$$

$$= \int_{-\infty}^{\infty} x(z) \cdot e^{-s(z \mp a)} \cdot dz$$

$$= \int_{-\infty}^{\infty} x(z) \cdot e^{-sz} \cdot e^{\pm sa} \cdot dz$$

$$L \{ x(t \pm a) \} = x(s) \cdot e^{\pm as}$$

⑦ frequency differentiation:

If  $L \{ x(t) \} = x(s)$  then,

$$L \{ t x(t) \} = -\frac{d}{ds} x(s)$$

proof: wkt,

$$x(s) = L \{ x(t) \} = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

Differentiating the above eq'n on both sides, w.r.t "s".

$$\frac{d}{ds} x(s) = \frac{d}{ds} \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot \left[ \frac{d}{ds} e^{-st} \right] \cdot dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot -t \cdot e^{-st} \cdot dt$$

$$= -t \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

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$$\frac{d}{ds} x(s) = \int_{-\infty}^{\infty} -t x(t) \cdot e^{-st} \cdot dt$$

$$= L \{ -t x(t) \}$$

$$= L^{-1} [ t x(s) ]$$

$$L \{ t x(t) \} = - \frac{d}{ds} x(s)$$

⑧ Frequency integration:

If  $L \{ x(t) \} = x(s)$  then,

$$L \left\{ \frac{1}{t} x(t) \right\} = \int_s^{\infty} x(s) \cdot ds$$

proof:

$$x(s) = L \{ x(t) \} = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

Integration above eq'n w.r.t "s" b/w limits  $s \rightarrow \infty$

$$\int_s^{\infty} x(s) \cdot ds = \int_s^{\infty} \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt \cdot ds$$

$$= \int_{-\infty}^{\infty} x(t) \left[ \int_s^{\infty} e^{-st} \cdot ds \right] \cdot dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[ \frac{e^{-st}}{-t} \right]_s^{\infty} \cdot dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[ 0 + \frac{e^{-st}}{t} \right] \cdot dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot \frac{e^{-st}}{t} \cdot dt$$

$$= L \left\{ \frac{1}{t} x(t) \right\}$$

$$\int_s^{\infty} x(s) \cdot ds = L \left\{ \frac{1}{t} x(t) \right\}$$

⑨ Scaling :

If  $L \{ x(t) \} = X(s)$  then,

$$L \{ x(at) \} = \frac{1}{|a|} X(s/a)$$

Proof :

$$L \{ x(at) \} = \int_{-\infty}^{\infty} x(at) \cdot e^{-st} \cdot dt$$

$$\text{put } at = z$$

$$dt = \frac{1}{a} \cdot dz$$

$$= \int_{-\infty}^{\infty} x(z) \cdot e^{-s(z/a)} \cdot \frac{1}{|a|} \cdot dz$$

$$= \frac{1}{|a|} \cdot \int_{-\infty}^{\infty} x(z) \cdot e^{-z(s/a)} \cdot dz$$

$$L \{ x(at) \} = \frac{1}{|a|} \cdot X\left(\frac{s}{a}\right)$$

Note: The above eq<sup>n</sup> transform is applicable for positive values of 'a'. If 'a' is negative then,

$$L \{ x(at) \} = -\frac{1}{|a|} X\left(\frac{s}{a}\right)$$

In general,  $L \{ x(at) \} = \frac{1}{|a|} X\left(\frac{s}{a}\right)$ .

### 10, periodicity :

If  $x(t) = x(t+nT)$  and  $x_1(t)$  be one period of  $x(t)$  and,  $L\{x_1(t)\} = \int_0^T x_1(t) \cdot e^{-st} \cdot dt$  then,

$$L\{x(t+nT)\} = \frac{1}{1-e^{-sT}} \int_0^T x_1(t) \cdot e^{-st} \cdot dt$$

proof :

$$L\{x(t+nT)\} = \int_0^{\infty} x(t+nT) \cdot e^{-st} \cdot dt$$

$$= \int_0^T x_1(t) \cdot e^{-st} \cdot dt + \int_T^{2T} x_1(t-T) e^{-s(t-T)} \cdot dt$$

$$+ \int_{2T}^{3T} x_1(t-2T) e^{-s(t-2T)} \cdot dt + \dots + \int_{pT}^{(p+1)T} x_1(t-pT) e^{-s(t-pT)} \cdot dt$$

$$= \sum_{p=0}^{\infty} \int_{pT}^{(p+1)T} x_1(t-pT) e^{-s(t-pT)} \cdot dt$$

$$= \sum_{p=0}^{\infty} \int_0^T x_1(t) e^{-st-pTs} = \int_0^T x_1(t) e^{-st} \left( \sum_{p=0}^{\infty} e^{-pTs} \right) \cdot dt$$

$$= \int_0^T x_1(t) e^{-st} \left( \sum_{p=0}^{\infty} e^{-sT} \right)^p \cdot dt$$

$$= \int_0^T x_1(t) e^{-st} \left( \frac{1}{1-e^{-sT}} \right) \cdot dt \quad \because \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} ; \alpha < 1$$

$$= \frac{1}{1-e^{-sT}} \int_0^T x_1(t) e^{-st}$$

$\therefore$  Hence proved.

Inverse Laplace transform : { By partial fraction expansion method }

There are three cases to find inverse Laplace transform by using partial fraction expansion method.

Case 1: When s-domain signal  $x(s)$  has distinct poles:

$$\text{Let } x(s) = \frac{k}{s(s+p_1)(s+p_2)}$$

by using partial fractions

$$x(s) = \frac{k_1}{s} + \frac{k_2}{(s+p_1)} + \frac{k_3}{(s+p_2)}$$

The residues  $k_1$ ,  $k_2$  &  $k_3$  are given by

$$k_1 = x(s) \cdot s \Big|_{s=0}$$

$$k_2 = x(s) \cdot (s+p_1) \Big|_{s=-p_1}$$

$$k_3 = x(s) \cdot (s+p_2) \Big|_{s=-p_2}$$

$$L^{-1}\{x(s)\} = L^{-1}\left\{\frac{k_1}{s} + \frac{k_2}{(s+p_1)} + \frac{k_3}{(s+p_2)}\right\}$$

$$\therefore x(t) = k_1 L^{-1}\left\{\frac{1}{s}\right\} + k_2 L^{-1}\left\{\frac{1}{s+p_1}\right\} + k_3 L^{-1}\left\{\frac{1}{s+p_2}\right\}$$

$$= k_1 u(t) + k_2 e^{-p_1 t} + k_3 e^{-p_2 t} \cdot u(t).$$

Case ii: When s-domain signal  $x(s)$  has multiple poles:

$$\text{Let } x(s) = \frac{k}{s(s+p_1)(s+p_2)^2}$$

by using partial fraction,

$$x(s) = \frac{k_1}{s} + \frac{k_2}{(s+p_1)} + \frac{k_3}{(s+p_2)^2} + \frac{k_4}{(s+p_2)}$$

$$k_1 = x(s) \cdot s \Big|_{s=0}$$

$$k_2 = x(s) \cdot (s+p_1) \Big|_{s=-p_1}$$

$$k_3 = x(s) \cdot (s+p_2)^2 \Big|_{s=-p_2}$$

$$k_4 = \frac{d}{ds} \left\{ x(s) \cdot (s+p_2) \right\} \Big|_{s=-p_2}$$

$$\therefore \mathcal{L}^{-1} \{ x(s) \} = \mathcal{L}^{-1} \left\{ \frac{k_1}{s} \right\} + \mathcal{L}^{-1} \left\{ \frac{k_2}{(s+p_1)} \right\} + \mathcal{L}^{-1} \left\{ \frac{k_3}{(s+p_2)^2} \right\} + \mathcal{L}^{-1} \left\{ \frac{k_4}{(s+p_2)} \right\}$$

$$\therefore x(t) = k \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} + k_2 \mathcal{L}^{-1} \left\{ \frac{1}{(s+p_1)} \right\} + k_3 \mathcal{L}^{-1} \left\{ \frac{1}{(s+p_2)^2} \right\} + \mathcal{L}^{-1} k_4 \left\{ \frac{1}{(s+p_2)} \right\}$$

$$x(t) = k_1 u(t) + k_2 e^{-p_1 t} u(t) + k_3 t e^{-p_2 t} u(t) + k_4 e^{-p_2 t} u(t)$$

$$\therefore \frac{1}{(s+a)^n} = t^n e^{-at}$$

$\therefore$  In general,

$$x(s) = \frac{1}{s(s+p_1)(s+p_2)^2} \text{ Then,}$$

$$x(s) = \frac{k_1}{s} + \frac{k_2}{(s+p_1)} + \frac{k_3}{(s+p_2)^2} + \frac{k_4}{(s+p_2)^{q-1}} + \dots + \frac{k(q-1)^2}{s+1}$$

$$k_{r2} = \frac{1}{r!} \frac{d^r}{ds^r} \left[ x(s) \cdot (s+p_2)^2 \right]$$

Where  $r = 1, 2, 3 \dots (q-1)$

Case iii: When s-domain signal  $x(s)$  has complex conjugate

poles:

$$\text{let } x(s) = \frac{k}{(s+p_1)(s^2+Bs+C)}$$

$$x(s) = \frac{k_1}{(s+p_1)} + \frac{k_2 s + k_3}{s^2 + Bs + C}$$

$$k_1 = x(s) \cdot (s+p_1) / s = -p_1$$

Arranging  $s^2 + Bs + C$  in  $(x+y)^2$  form

$$x(s) = \frac{k_1}{(s+p_1)} + \frac{k_2 s + k_3}{s^2 + \frac{2Bs}{2} + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + C}$$

$$\text{put } a = \frac{b}{2} \quad c - \left(\frac{b^2}{4}\right) = \omega_0^2$$

$$X(s) = \frac{k_1}{s - p_1} + \frac{k_2 s + k_3}{\left(s - \frac{b}{2}\right)^2 + \left(c - \left(\frac{b}{2}\right)^2\right)}$$

$$X(s) = \frac{k_1}{s - p_1} + \frac{k_2 \left(s + a + \frac{k_3 - a}{k_2}\right)}{(s + a)^2 + \omega_0^2}$$

$$= \frac{k_1}{(s - p_1)} + k_2 \cdot \frac{[s + a + k_4]}{(s + a)^2 + \omega_0^2} \quad \because \frac{k_3 - a}{k_2} = k_4$$

$$X(s) = \frac{k_1}{s - p_1} + k_2 \cdot \frac{(s + a)}{(s + a)^2 + \omega_0^2} + k_5 \cdot \frac{\omega_0}{(s + a)^2 + \omega_0^2} \quad \left[ \because \frac{k_2 k_4}{\omega_0} = k_5 \right]$$

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{k_1}{s - p_1}\right\} + \mathcal{L}^{-1}\left\{k_2 \cdot \frac{(s + a)}{(s + a)^2 + \omega_0^2}\right\} + \mathcal{L}^{-1}\left\{k_5 \cdot \frac{\omega_0}{(s + a)^2 + \omega_0^2}\right\}$$

$$\therefore x(t) = k_1 \cdot e^{-p_1 t} u(t) + k_2 e^{-at} \cos \omega_0 t u(t) + k_5 e^{-at} \sin \omega_0 t u(t)$$

Laplace transform of unit step function:

Laplace transform of unit step,

$$\mathcal{L}\{x(t)\} = X(s) = \int_0^{\infty} x(t) \cdot e^{-st} \cdot dt$$

$$\text{But } x(t) = u(t) = \begin{cases} 1 & ; t \geq 0 \\ 0 & ; t < 0 \end{cases}$$

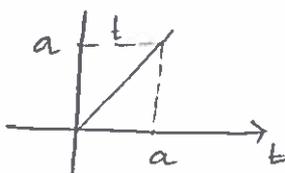
$$\therefore X(s) = \int_0^{\infty} u(t) \cdot e^{-st} \cdot dt$$

$$= \int_0^{\infty} 1 \times e^{-st} \cdot dt$$

$$\therefore X(s) = \left[ \frac{-e^{-st}}{-s} \right]_0^{\infty} = \left[ 0 + \frac{1}{s} \right]$$

$$\therefore X(s) = \frac{1}{s}$$

Ramp function:



$$x(t) = \begin{cases} t; & t > 0 \\ 0; & t < 0 \end{cases}$$

$$x(t) = t u(t)$$

$$X(t) = \begin{cases} t; & 0 \leq t \leq a \\ 0; & t > a \end{cases}$$

$$\therefore L\{X(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

$$= \int_0^a t \cdot e^{-st} \cdot dt$$

$$= \left[ t \frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^a$$

$$= \left[ \frac{s^{-s} a}{-s} - \frac{e^{-at}}{s^2} - 0 + \frac{1}{s^2} \right]$$

$$= \frac{1}{s^2} \left[ 1 - e^{-as} - as e^{-sa} \right]$$

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$$x(s) = \frac{1}{s} \left[ 1 - e^{-as} (1+as) \right]$$

⇒ The z-transform of a  $x(n)$  is denoted by  $X(z)$  and defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

where "z" is called Complex Variable

⇒  $x(n)$  &  $X(z)$  are called z-transform pair represented as

$$x(n) \longleftrightarrow X(z)$$

⇒ The purpose learning z-transform is to analyze the discrete time signal and systems, digital filters and synthesis of digital filter systems.

→ For any input sequence, the z-transform is complex, i.e. contains real & imaginary parts

⇒ Distinction between laplace fourier & z-transform:-

z-transform is as  $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$  → ①

where z is defined  $z = re^{j\omega}$

where 'r' is magnitude of z i.e. on  $|z|$

and " $\omega$ " is the angle of z

Substituting the value of 'z' in ①

$$\textcircled{1} \rightarrow X(z) = \sum_{n=-\infty}^{\infty} x(n) (ze^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [x(n) r^{-n}] e^{-j\omega n} \rightarrow \textcircled{2}$$

Fourier transform is given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \rightarrow \textcircled{3}$$

Comparing eq'n ② & ③ we can find that

$X(z)$  indicates Fourier transform of  $[x(n) r^{-n}]$

Let  $X(z)$  of eq ② be evaluated on unit circle

(i.e. radius of circle should always 1) then,  $|z| = r = 1$

$$\text{i.e. } r^{-n} = 1$$

$$\therefore X(z)/z = e^{j\omega} = \sum_{n=-\infty}^{\infty} (x(n) e^{-j\omega n})$$

$$\therefore X(z) = X(\omega)$$

This is the relationship between Fourier transform

& Z-Transform

Laplace transform of input signal  $x(t)$  is defined

as

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt \rightarrow \textcircled{1}$$

But  $s = -j\omega$  ; substituting 's' value in  $\textcircled{1}$

$$X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{(-j\omega)t} \cdot dt \rightarrow \textcircled{2}$$

by definition of fourier transform of  $x(t)$  is given by

$$F\{x(t)\} = X(-j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt \rightarrow \textcircled{3}$$

Comparing eq'n  $\textcircled{2}$  &  $\textcircled{3}$  we can say that by substituting  $\omega = 0$  in Laplace transform eq'n we get fourier transform of continuous time signal.

$$\therefore x(j\omega) = X(s) / s = j\omega$$

Region of Convergence      z-transform:

Since, z-transform is an infinite power series, it exists only for those values of z for which the series converges.

The Roc of  $x(z)$  is the set of all values of z for which  $x(z)$  attains a finite values.

## Properties:

1. The Roc is a ring or disc in the  $z$ -plane centered at origin

2. The Roc doesn't contains any poles

3. if  $x(n)$  is a finite duration casual sequence then Roc is the entire  $z$ -plane except at  $z=0$

Let  $x(n)$  be finite duration signal with  $n$ -samples defined in the range  $0 \leq n \leq (N-1)$

$$\therefore x(n) = \{ x(0), x(1), x(2), \dots, x(N-1) \}$$

$$\therefore X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$$

$$= x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + \dots + x(N-1)z^{-(N-1)}$$

$$= x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \dots + \frac{x(N-1)}{z^{(N-1)}}$$

When  $z=0$ ; all the term in the eq'n except the first term becomes infinite

Hence  $X(z)$  exists for all values of  $z$  except  $z=0$

Therefore Roc for finite  $z$ -plane except  $z=0$

4. If  $x(n)$  is a finite duration anti-casual sequence (non-casual) then Roc is entire  $z$ -plane except at  $z=\infty$

$x(n)$  be finite duration signal with  $n$ -samples defined in the range  $-(N-1) \leq n \leq 0$

$$\therefore X(z) = \sum_{n=-(N-1)}^0 x(n) z^{-n}$$

$$X(z) = x(-(N-1)) z^{(N-1)} + \dots + (x(-2) z^2 + x(-1) z^1 + x(0) z^0$$

When  $z = \infty$  all the terms of above eq'n except the last term become infinite. Hence  $X(z)$  exists for all the values of  $z$  except  $z = \infty$

$\therefore$  Roc of  $X(z)$  entire  $z$ -plane except  $z = \infty$

Property 5: The Roc must be Connected Region.

Properties:

1) linearity:

$$Z \{ a_1 x_1(n) + a_2 x_2(n) \} = a_1 X_1(z) + a_2 X_2(z)$$

Proof:

$$X_1(z) = Z \{ x_1(n) \} = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n}$$

$$X_2(z) = Z \{ x_2(n) \} = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$\begin{aligned}
 \mathcal{Z}\{a_1 x_1(n) + a_2 x_2(n)\} &= \sum_{n=-\infty}^{\infty} [a_1 x_1(n) + a_2 x_2(n)] z^{-n} \\
 &= \sum_{n=-\infty}^{\infty} (a_1 x_1(n) z^{-n} + a_2 x_2(n) z^{-n}) \\
 &= \sum_{n=-\infty}^{\infty} a_1 x_1(n) z^{-n} + \sum_{n=-\infty}^{\infty} a_2 x_2(n) z^{-n} \\
 &= a_1 x_1(z) + a_2 x_2(z)
 \end{aligned}$$

(2) Shifting property:

Let  $\mathcal{Z}\{x(n)\} = X(z)$  then,

$$\mathcal{Z}\{x(n-m)\} = z^{-m} X(z) \quad \& \quad \mathcal{Z}\{x(n+m)\} = z^m X(z)$$

proof:

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(p) z^{-p(m)}$$

$$= \sum_{n=-\infty}^{\infty} x(p) z^{-p} z^{-m}$$

$$= z^{-m} \sum_{n=-\infty}^{\infty} x(p) z^{-p} = z^{-m} X(z)$$

$$\mathcal{Z}\{x(n+m)\} = \sum_{n=-\infty}^{\infty} x(n+m) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(p) z^{-(p-m)}$$

$$= \sum_{n=-\infty}^{\infty} x(p) z^{-p} \cdot z^m$$

$$= z^m \sum_{n=-\infty}^{\infty} x(p) z^{-p}$$

Differentiation z-domain:

let  $x(n) \xrightarrow{ZT} X(z)$ , ROC:  $r_1 < |z| < r_2$  then

$n x(n) \xrightarrow{ZT} z \frac{d}{dz} X(z)$ , ROC:  $r_1 < |z| < r_2$

Proof: Wk1,  $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

Differentiating both sides w.r to  $z$

$$\frac{d}{dz} X(z) = \frac{d}{dz} \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \frac{d}{dz} z^{-n}$$

$$= - \sum_{n=-\infty}^{\infty} n x(n) z^{-n-1}$$

$$= - z^{-1} \sum_{n=-\infty}^{\infty} [n x(n)] z^{-n}$$

$$\frac{d}{dz} X(z) = - z^{-1} \cdot z \{n x(n)\}$$

$$\frac{d}{dz} X(z) = -\frac{1}{z} \cdot z \{n x(n)\}$$

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(or)

$$-z \frac{d}{dz} X(z) = z \{n x(n)\};$$

Hence proved.

⇒ IZT by using power series expansion / long division method:

⇒ It is calculated based on Roc

⇒ If Roc is right sided sequence ( $|z| > 2$ ) then  $x(n)$  consists of +ve time sequence

∴ Divide the numerator by denominator get Co-efficient consisting of -ve powers of 'z'

⇒ If Roc is left sided sequence ( $|z| < 2$ ) then  $x(n)$  consists of -ve time sequence

∴ Divide the numerator by denominator to get Co-efficient consists of +ve power of 'z'

Ros → (+ve time sequence) →  $\frac{N(z)}{D(z)} \rightarrow$  -ve power of z

Ros → (-ve time sequence) →  $\frac{N(z)}{D(z)} \rightarrow$  +ve power of z

IzT by using partial fraction expansion method:

1. Arrange the given  $x(z)$  as  $\frac{x(z)}{z} = \frac{\text{Numerator poles nomia}}{(z-p_1)(z-p_2)(z-p_3)\dots(z-p_n)}$

$$2. \frac{x(z)}{z} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \frac{A_3}{z-p_3} + \dots + \frac{A_n}{z-p_n} \rightarrow \textcircled{1}$$

Where  $A_1, A_2, A_3, \dots, A_n$  are arbitrary constant.

Where

$$A_k = (z-p_k) \frac{x(z)}{z} \Big|_{z=p_k} \quad k=1, 2, 3, 4, \dots, n$$

If  $\frac{x(z)}{z}$  has poles of multiplicity "n" i.e

$$\frac{x(z)}{z} = \frac{\text{Numerator polynomial}}{(z-p)^n}$$

$$= \frac{A_1}{(z-p)} + \frac{A_2}{(z-p)^2} + \dots + \frac{A_n}{(z-p)^n}$$

$$\text{Where } A_k = \frac{1}{(n-k)!} \cdot \frac{d^{n-k}}{dz^{n-k}} \left\{ (z-p)^n \cdot \frac{x(z)}{z} \right\} \Big|_{z=p}$$

$$k=1, 2, 3, 4, \dots, n$$

Equation (1) can be written as

$$X(z) = \frac{A_1 z}{(z-p_1)} + \frac{A_2 z}{(z-p_2)} + \dots + \frac{A_n z}{(z-p_n)}$$

$$= \frac{A_1}{1-p_1 z^{-1}} + \frac{A_2}{1-p_2 z^{-1}} + \dots + \frac{A_n}{1-p_n z^{-1}}$$

i.e., all the terms in above step are of the form

$$\frac{A_k}{1-p_k z^{-1}}$$

Depending on Roc following standard z-transform pairs must be used.

$$p_k^n u(n) \xrightarrow{zT} \frac{1}{1-p_k z^{-1}}, \quad \text{Roc: } |z| > |a| \text{ casual Sequence}$$

$$-(p_k)^n u(-n-1) \xrightarrow{zT} \frac{1}{1-p_k z^{-1}}; \quad |z| < |a| \text{ i.e. non casual}$$